Problem Set 3 (Solutions) (Due 1/29/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work.

P1. Let U be a domain. Prove that if f is a real-valued function analytic on U, then f is constant on U.

Proof. If $f(z) = u(x, y) + iv(x, y)$, then $v(x, y) = 0$ by assumption. We show that $f'(z) = 0$ on U. Since f is analytic on U, the Cauchy-Riemann equations are satisfied on U. But then $u_x = v_y = 0$. Hence, $f'(z) = u_x(x, y) + iv_x(x, y) = 0$. But U is a domain so this implies that f is constant, according to a result from class. ☕

P2. Let $f(z) = u + iv$ be a complex-valued function defined on an open set $U \subseteq \mathbb{C}$. Suppose that the first-order partial derivatives of $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$ exist and are continuous on U. Define the differential operators[1]

$$\frac{\partial f}{\partial z} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Note: we also define $\frac{\partial f}{\partial x} \overset{\text{def}}{=} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, and similarly for $\frac{\partial}{\partial y}$.

(a) Show that f is analytic on U if and only if and only if $\frac{\partial f}{\partial \overline{z}} = 0$.

(b) If f is analytic on U, show that $f' = \frac{\partial f}{\partial z}$.

Proof. (a) First, suppose that f is analytic on U. Then the real and imaginary parts of f satisfy the Cauchy-Riemann equations. Hence,

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\right) && \text{(by the CR equations)} \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\right) \\
&= 0.
\end{aligned}$$

[1] These operators are inspired by the following formal (meaningless) calculation. Consider the functions $x(z, \overline{z}) = \frac{z+\overline{z}}{2}$ and $y(z, \overline{z}) = \frac{z-\overline{z}}{2i}$ (note $z = x + iy$) and formally apply the chain rule

$$\begin{aligned}
\frac{\partial}{\partial z} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{\partial}{\partial x}\frac{1}{2} + \frac{\partial}{\partial y}\frac{1}{2i}, \\
\frac{\partial}{\partial \overline{z}} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{\partial}{\partial x}\frac{1}{2} - \frac{\partial}{\partial y}\frac{1}{2i}.
\end{aligned}$$

Conversely, suppose that $\frac{\partial f}{\partial \bar{z}} = 0$. Since we already know that the real and imaginary parts of f have continuous first-order partial derivatives on U, it is sufficient to show that the Cauchy-Riemann equations are satisfied. We compute

$$\begin{aligned}0 = \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{1}{2}i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).\end{aligned}$$

Comparing the real and imaginary parts of each side of the equation, we conclude that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. These are precisely the CR equations, so this proves the claim.

(b) If f is analytic, then the Cauchy-Riemann equations are satisfied and $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. Hence, we have

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{2}i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) + \frac{1}{2}i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right) && \text{(by the CR equations)} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= f'(z).\end{aligned}$$

This proves the claim.

P3. (Optional.)[2] Let U be an open set and let f be a function that is continuous on U with the property

$$e^{f(z)} = z, \ z \in U.$$

(a) Show that f is analytic on U.[3]

(b) Consider the function $f(z) = \log z \stackrel{\text{def}}{=} \ln|z| + i\arg z, \ |z| > 0, \ -\pi < \arg z \leq \pi$. What, if anything, does part (a) tell you about f? Justify your conclusions.

Proof. (a) Let $z \in U$. Since f is continuous, we have $\lim_{w\to z} f(w) = f(z)$. Hence, since also the exponential function is entire,

$$\lim_{w\to z} \frac{e^{f(w)} - e^{f(z)}}{f(w) - f(z)} = \lim_{vv\to f(z)} \frac{e^{v} - e^{f(z)}}{v - f(z)} = e^{f(z)}.$$

[2]This is worth 3 points extra credit, added to your score on this assignment.
[3]This shows that a continuously defined logarithm on an open set is automatically analytic.

Thus, we obtain

$$\begin{aligned}\lim_{w\to z}\frac{f(w)-f(z)}{w-z} &= \lim_{w\to z}\frac{f(w)-f(z)}{e^{f(w)}-e^{f(z)}}\\ &= \lim_{w\to z}\frac{1}{\frac{e^{f(w)}-e^{f(z)}}{f(w)-f(z)}}\\ &= \frac{1}{\lim_{w\to z}\frac{e^{f(w)}-e^{f(z)}}{f(w)-f(z)}}\\ &= \frac{1}{e^{f(z)}}\\ &= \frac{1}{z}.\end{aligned}$$

Since the limit exists, we have shown that f is analytic on U. Moreover, we have shown that $f'(z) = \frac{1}{z}$.

(b) Trick question, (a) tells you nothing about this functions – because f is not continuous on its domain ($\mathbb{C}\setminus\{0\}$). Specifically, it is not continuous at any point on the negative real axis. For instance, consider $z = -1$. Approaching -1 perpendicularly from the upper half plane, the limit is πi, while approaching perpendicularly from the lower half plane, the limit is $-\pi i$.

P4. Find all solutions to the equation $e^{2z} - 2ie^z = 1$.

Solution. Use the quadratic formula or factor to get $e^z = i$. Then the solutions are $z = \log i = \ln|i| + i\arg i = \ln 1 + i(\pi/2 + 2k\pi) = i(\pi/2 + 2k\pi)$ where $k \in \mathbb{Z}$.

P5. (a) Find real valued functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that $e^z = u(x, y) + iv(x, y)$.

(b) Find real valued functions $U, V : \mathbb{R}^2 \to \mathbb{R}$ such that $\cos z = U(x, y) + iV(x, y)$.

(c) Show that $e^{\overline{z}} = \overline{e^z}$ and $\cos\overline{z} = \overline{\cos z}$.

Solution. (a) Using the definition of the complex exponential function and Euler's formula:

$$\begin{aligned}e^z &= e^x e^{-y}\\ &= e^x\cos y + ie^x\sin y.\end{aligned}$$

So $u(x, y) = e^x\cos y$ and $v(x, y) = e^x\sin y$.

(b) Using the definition of the complex-valued cosine function and the definition of the complex exponential function:

$$\begin{aligned}2\cos z &= e^{iz} + e^{-iz}\\ &= e^{ix-y} + e^{-ix+y}\\ &= e^{-y}e^{ix} + e^{y}e^{-ix}\\ &= e^{-y}\cos x + ie^{-y}\sin x + e^{y}\cos(-x) + ie^{y}\sin(-x)\\ &= (e^{y}+e^{-y})\cos x - i(e^{y}-e^{-y})\sin x\\ &= 2\cosh y\cos x - 2i\sinh y\sin x.\end{aligned}$$

Hence, (after dividing by 2) $U(x, y) = \cosh y\cos x$ and $V(x, y) = -\sinh y\sin x$. Do you recognize these functions from the chapter 2 lectures? Take a look at the Ch2 notes, pg 19.

(c) These follow easily from (a) and (b) and the fact that $\sin x$ and $\cos x$ are odd and even functions, respectively. First, using (a) we have

$$\begin{aligned}\overline{e^z} &= e^x \cos y - ie^x \sin y \\ &= e^x \cos(-y) + ie^x \sin(-y) \\ &= e^{\overline{z}}.\end{aligned}$$

Second, using (b) we have

$$\begin{aligned}\overline{\cos z} &= \cosh y \cos x + i \sinh y \sin x \\ &= \cosh y \cos(-x) - i \sinh x \sin(-x) \\ &= \cos \overline{z}.\end{aligned}$$

P6. Determine the points at which the following functions are analytic:

(a) $e^{\overline{z}}$;

(b) $\cos \overline{z}$.

Proof. (a) The preceding problem is screaming at us to use the Cauchy-Riemann equations. Using P5 (a) and (c), we have $e^{\overline{z}} = \overline{e^z} = u(x,y) - iv(x,y) = e^x \cos y - ie^x \sin y$. The Cauchy-Riemann equations yield $e^x \cos y = -e^x \cos y$ and $-e^x \sin y = e^x \sin y$. Since $e^x > 0$, these reduce to $\cos y = -\cos y$ and $\sin y = -\sin y$. Hence, $\cos y = 0$ and $\sin y = 0$. There are no values of y satisfying both equations. Hence, the Cauchy-Riemann equations are satisfied nowhere and we can conclude that $e^{\overline{z}}$ is analytic nowhere.

(b) Similarly, from P5 (b) and (c) we have $\sin \overline{z} = \overline{\sin z} = \cosh y \cos x + i \sinh y \sin x$. The Cauchy-Riemann equations are then

$$-\cosh y \sin x = \cosh y \sin x \quad \text{and} \quad \sinh y \cos x = -\sinh y \cos x.$$

Hence, $\cosh y \sin x = 0$ and $\sinh y \cos x = 0$. Since $\cosh y$ has no zeros, we obtain $\sin x = 0$ so that $x = k\pi$, $k \in \mathbb{Z}$. Hence, $\cos x \neq 0$, which implies that $\sinh y = 0$. Hence, $y = 0$. So, the Cauchy-Riemann equations hold only at the points $z = k\pi$ for $k \in \mathbb{Z}$. But these points are all isolated – there is not neighborhood of any point throughout which $\sin \overline{z}$ is analytic. Thus, $\sin \overline{z}$ is nowhere analytic.

Note: it is possible that $\cos \overline{z}$ exists at the points $z = k\pi$, but analyicity is a stronger requirement.

P7. Let $z \in \mathbb{C}$.

(a) Prove that $\cancel{1^z}$ $|1^z|$ is single-valued if and only if $\operatorname{Im} z = 0$.

(b) Find a necessary and sufficient condition for $|i^{iz}|$ to be single-valued.

(c) **(Optional.)**[4] Show by counterexample that the statement is false: 1^z is single-valued if and only if $\operatorname{Im} z = 0$.

[4] Worth 1 point extra credit, added to your score on this assignment.

Proof. (a) A calculator shows:

$$1^z = e^{(x+iy)\log 1} = e^{(x+iy)2k\pi i} = e^{-y2k\pi}e^{x2k\pi i}.$$

Since $e^{-y2k\pi} \in \mathbb{R} \setminus \{0\}$ and $x2k\pi i$ is purely imaginary, we see that we have written 1^z is exponential form (polar form). Hence, $|1^z| = e^{-y2k\pi}$. Since k can be any integer, this is clearly single-valued if and only if $\operatorname{Im} z = y = 0$.

(b) We claim that $\left|i^{iz}\right|$ is single-valued if and only if $\operatorname{Re} z = 0$. A computation shows:

$$i^{iz} = e^{i(x+iy)\log i} = e^{(ix-y)(\pi/2+2k\pi)i} = e^{-x(\pi/2+2k\pi)}e^{-y(\pi/2+2k\pi)i}.$$

Now this has been written in exponential form since $e^{-x(\pi/2+2k\pi)} \in \mathbb{R} setminus\{0\}$ and $-y(\pi/2 + 2k\pi)i$ is purely imaginary. Hence $\left|i^{iz}\right| = e^{-x(\pi/2+2k\pi)}$ which depends on the integer $k \in \mathbb{Z}$ and hence is single-valued if and only if $\operatorname{Re} z = x = 0$.

(c) Take $z = 1/2$ so that $\operatorname{Im} z = 0$. Then $1^{1/2} = e^{\frac{1}{2}\log 1} = e^{k\pi i} = \pm 1$.